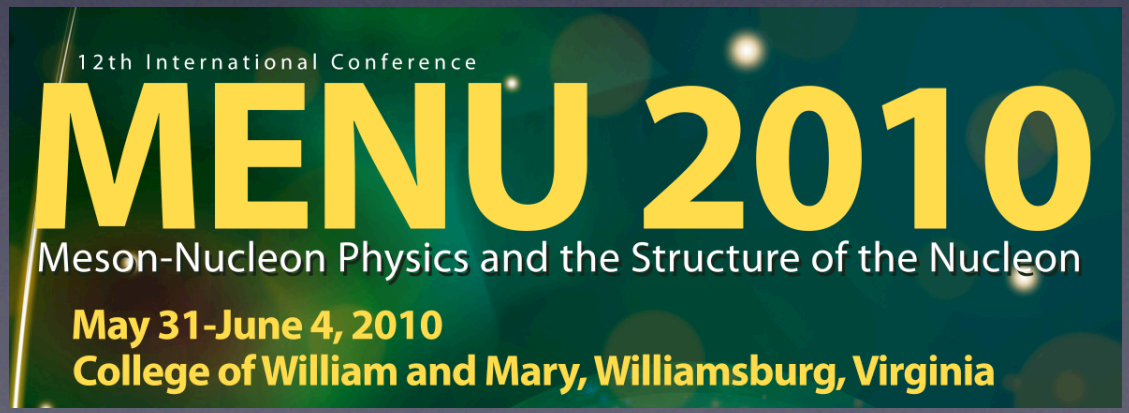
A New Approach for Delta Form Factors

Christopher Aubin
College of William & Mary

w/ Kostas Orginos



Calculating Form Factors

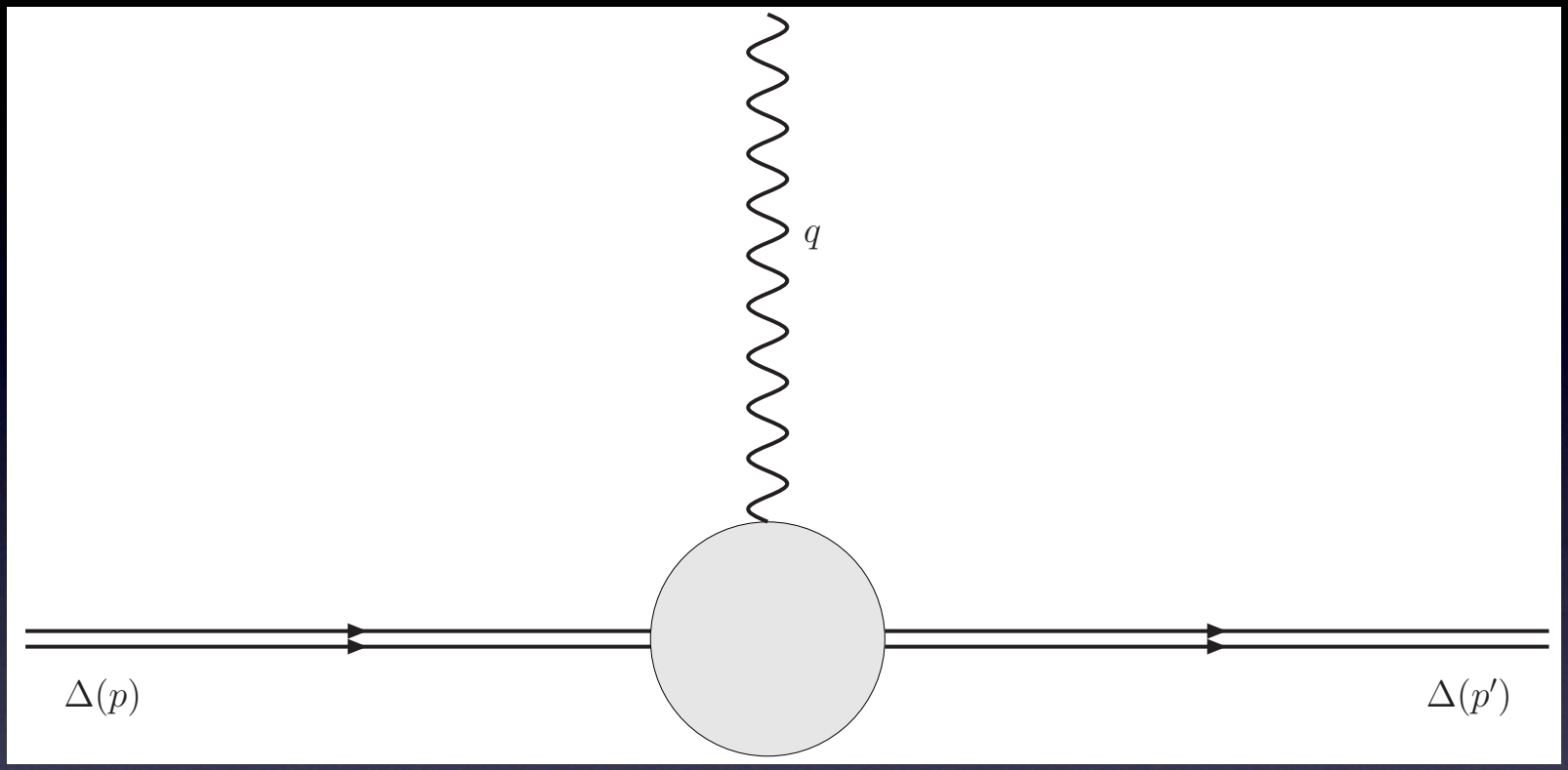
Difficulties in lattice calculations:

Finite Volume, disconnected diagrams...

Extracting ground states

Generalized Pencil-of-Function Method

The electromagnetic form factors for the Delta:



$$\langle \Delta(p') | V^\mu | \Delta(p) \rangle = \bar{u}_\alpha(p') \Gamma^{\alpha\mu\beta} u_\beta(p)$$

$\Delta(1232)$

The electromagnetic form factors for the Delta:

$$\begin{aligned}\Gamma^{\alpha\beta\mu}_{\gamma\Delta\Delta}(p',p) &= -e\left\{e_\Delta\, F_1^*(Q^2)\, g^{\alpha\beta}\,\gamma^\mu\right.\\ &+ \frac{i}{2M_\Delta}\left[F_2^*(Q^2)\, g^{\alpha\beta} + F_4^*(Q^2)\,\frac{q^\alpha q^\beta}{(2M_\Delta)^2}\right]\sigma^{\mu\nu}q_\nu\\ &+ \left.\frac{F_3^*(Q^2)}{(2M_\Delta)^2}\left[q^\alpha q^\beta\gamma^\mu - \frac{1}{2}q\cdot\gamma\,(g^{\alpha\mu}q^\beta + g^{\beta\mu}q^\alpha)\right]\right\}\end{aligned}$$

At $Q^2=0$:

$\Delta(1232)$

The electromagnetic form factors for the Delta:

$$\Gamma^{\alpha\beta\mu}_{\gamma\Delta\Delta}(p',p) = -e\left\{e_\Delta F_1^*(Q^2)\, g^{\alpha\beta}\,\gamma^\mu + \frac{i}{2M_\Delta}\left[F_2^*(Q^2)\, g^{\alpha\beta} + F_4^*(Q^2)\,\frac{q^\alpha q^\beta}{(2M_\Delta)^2}\right]\sigma^{\mu\nu}q_\nu + \frac{F_3^*(Q^2)}{(2M_\Delta)^2}\left[q^\alpha q^\beta\gamma^\mu - \frac{1}{2}q\cdot\gamma\,(g^{\alpha\mu}q^\beta + g^{\beta\mu}q^\alpha)\right]\right\}$$

At $Q^2=0$:

Charge

$\Delta(1232)$

The electromagnetic form factors for the Delta:

$$\Gamma^{\alpha\beta\mu}_{\gamma\Delta\Delta}(p',p) = -e\left\{e_\Delta F_1^*(Q^2)\, g^{\alpha\beta}\,\gamma^\mu + \frac{i}{2M_\Delta}\left[F_2^*(Q^2)\, g^{\alpha\beta} + F_4^*(Q^2)\,\frac{q^\alpha q^\beta}{(2M_\Delta)^2}\right]\sigma^{\mu\nu}q_\nu + \frac{F_3^*(Q^2)}{(2M_\Delta)^2}\left[q^\alpha q^\beta\gamma^\mu - \frac{1}{2}q\cdot\gamma\,(g^{\alpha\mu}q^\beta + g^{\beta\mu}q^\alpha)\right]\right\}$$

At $Q^2=0$:

Charge

Magnetic Dipole

$\Delta(1232)$

The electromagnetic form factors for the Delta:

$$\Gamma^{\alpha\beta\mu}_{\gamma\Delta\Delta}(p',p) = -e\left\{e_\Delta F_1^*(Q^2)\, g^{\alpha\beta}\,\gamma^\mu + \frac{i}{2M_\Delta}\left[F_2^*(Q^2)\, g^{\alpha\beta} + F_4^*(Q^2)\,\frac{q^\alpha q^\beta}{(2M_\Delta)^2}\right]\sigma^{\mu\nu}q_\nu + \frac{F_3^*(Q^2)}{(2M_\Delta)^2}\left[q^\alpha q^\beta\gamma^\mu - \frac{1}{2}q\cdot\gamma\,(g^{\alpha\mu}q^\beta + g^{\beta\mu}q^\alpha)\right]\right\}$$

At $Q^2=0$:

Charge

Electric Quadrupole

Magnetic Dipole

$\Delta(1232)$

The electromagnetic form factors for the Delta:

$$\Gamma^{\alpha\beta\mu}_{\gamma\Delta\Delta}(p',p) = -e\left\{e_\Delta F_1^*(Q^2)\, g^{\alpha\beta}\,\gamma^\mu + \frac{i}{2M_\Delta}\left[F_2^*(Q^2)\, g^{\alpha\beta} + F_4^*(Q^2)\,\frac{q^\alpha q^\beta}{(2M_\Delta)^2}\right]\sigma^{\mu\nu}q_\nu + \frac{F_3^*(Q^2)}{(2M_\Delta)^2}\left[q^\alpha q^\beta\gamma^\mu - \frac{1}{2}q\cdot\gamma\left(g^{\alpha\mu}q^\beta + g^{\beta\mu}q^\alpha\right)\right]\right\}$$

At $Q^2=0$:

Charge

Electric Quadrupole

Magnetic Dipole

Magnetic Octupole

From the PDG

$$\mu_{\Delta^{++}} = (5.6 \pm 1.9)\mu_N$$

$$\mu_{\Delta^{+}} = (2.7 \pm 3.5)\mu_N$$

Even getting the magnetic moment experimentally is difficult for the Delta

The lattice is essential in determining the form factors in the absence of experimental data

On the lattice, we calculate n-point correlators:

$$C^{3pt}(t_i, t, t_f, \mathbf{p}_i, \mathbf{p}_f) = FT[\langle 0|\chi(t_f)J_\mu(t)\chi(t_i)|0\rangle]$$

For $t_f \gg t \gg t_i$:

$$C^{3pt}(t_i, t, t_f, \mathbf{p}_i, \mathbf{p}_f) \quad \rightarrow \quad Z(\mathbf{p}_i, \mathbf{p}_f)e^{-E_f(t_f - t)}e^{-E_i(t - t_i)}$$
$$\times \langle \Delta(p_f)|J_\mu(0)|\Delta(p_i)\rangle$$

On the lattice, we calculate n-point correlators:

$$C^{3pt}(t_i, t, t_f, \mathbf{p}_i, \mathbf{p}_f) = FT[\langle 0|\chi(t_f)J_\mu(t)\chi(t_i)|0\rangle]$$

For $t_f \gg t \gg t_i$:

$$C^{3pt}(t_i, t, t_f, \mathbf{p}_i, \mathbf{p}_f) \quad \rightarrow \quad Z(\mathbf{p}_i, \mathbf{p}_f)e^{-E_f(t_f-t)}e^{-E_i(t-t_i)}$$
$$\times\langle\Delta(p_f)|J_\mu(0)|\Delta(p_i)\rangle$$

And Z includes overlap factors that also arise in the two-point functions

$$C^{2pt}(t_i, t_f, \mathbf{p}) \rightarrow A' e^{-(t_f-t)E}$$

On the lattice, we calculate n-point correlators:

$$C^{3pt}(t_i, t, t_f, \mathbf{p}_i, \mathbf{p}_f) = FT[\langle 0|\chi(t_f)J_\mu(t)\chi(t_i)|0\rangle]$$

For $t_f \gg t \gg t_i$:

$$C^{3pt}(t_i, t, t_f, \mathbf{p}_i, \mathbf{p}_f) \quad \rightarrow \quad Z(\mathbf{p}_i, \mathbf{p}_f)e^{-E_f(t_f-t)}e^{-E_i(t-t_i)} \times \langle \Delta(p_f)|J_\mu(0)|\Delta(p_i)\rangle$$

And Z includes overlap factors that also arise in the two-point functions

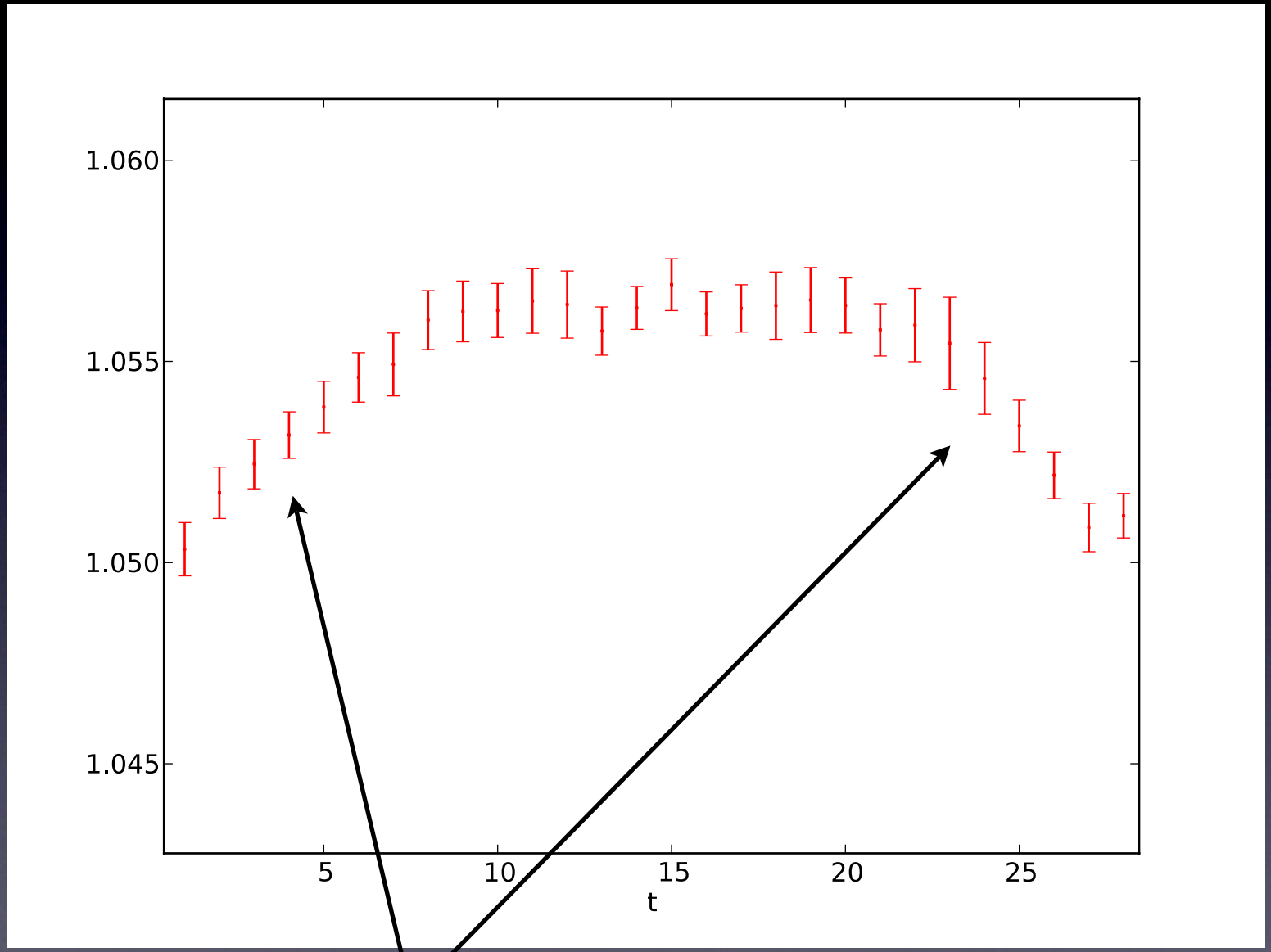
$$C^{2pt}(t_i, t_f, \mathbf{p}) \rightarrow A' e^{-(t_f-t)E}$$

$$\left[\frac{C^{3pt}}{C^{2pt}}\right]_{\text{appropriate}} \rightarrow \langle \Delta(p_f)|J_\mu(0)|\Delta(p_i)\rangle$$

proton

$$G_E(p^2) = F_1(p^2) - \frac{p^2}{2m} F_2(p^2)$$

$$G_E(p^2 = 0)$$

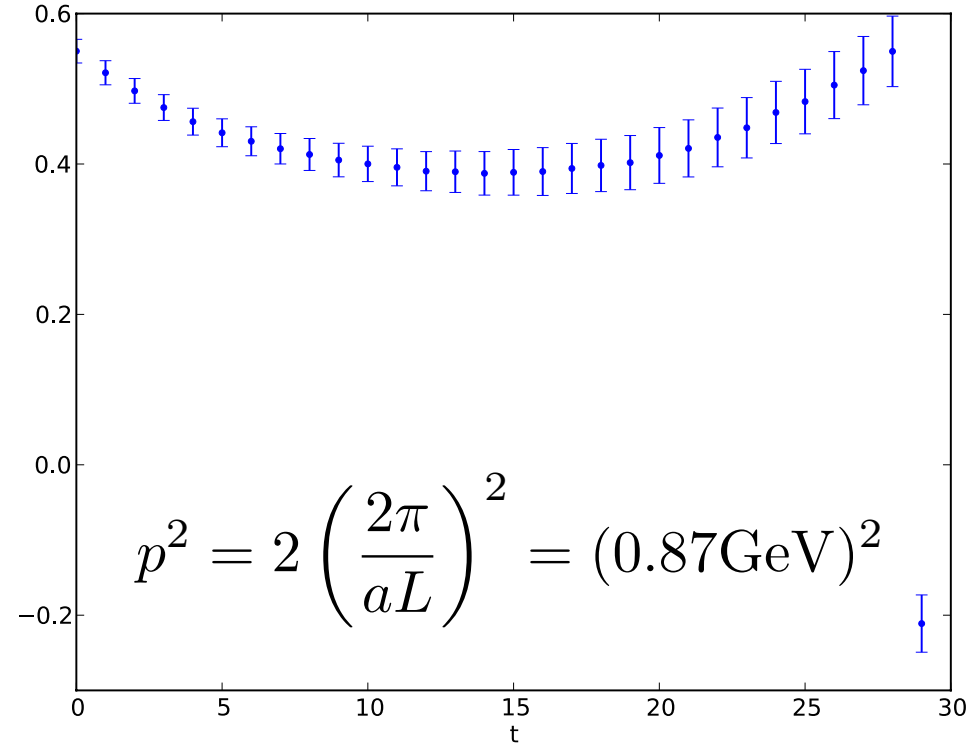


Excited state contamination

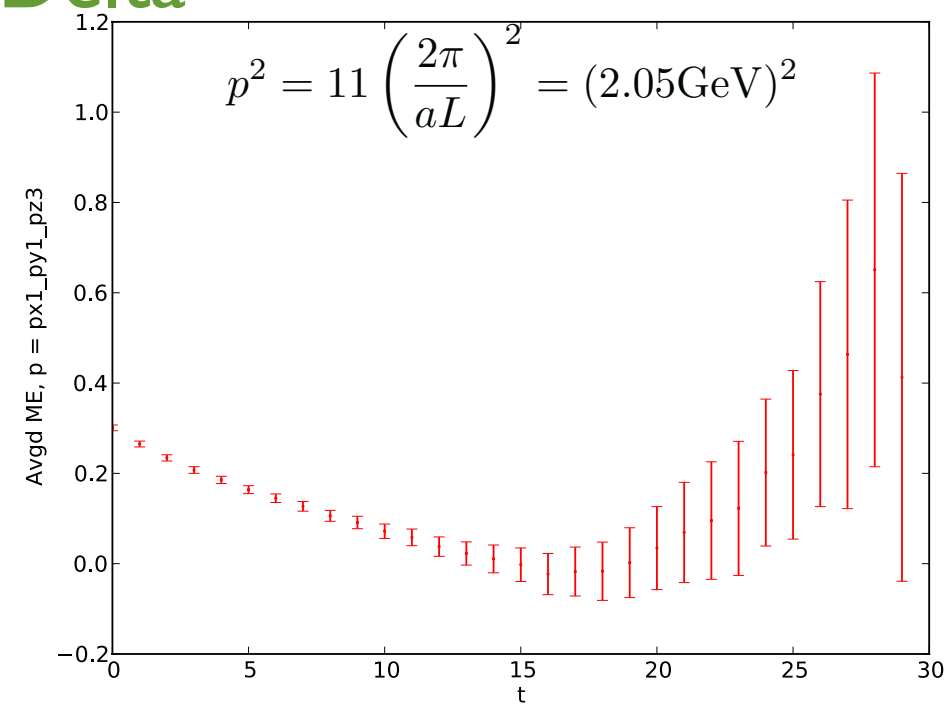
$G_E(p^2)$

Higher momenta

proton



Delta

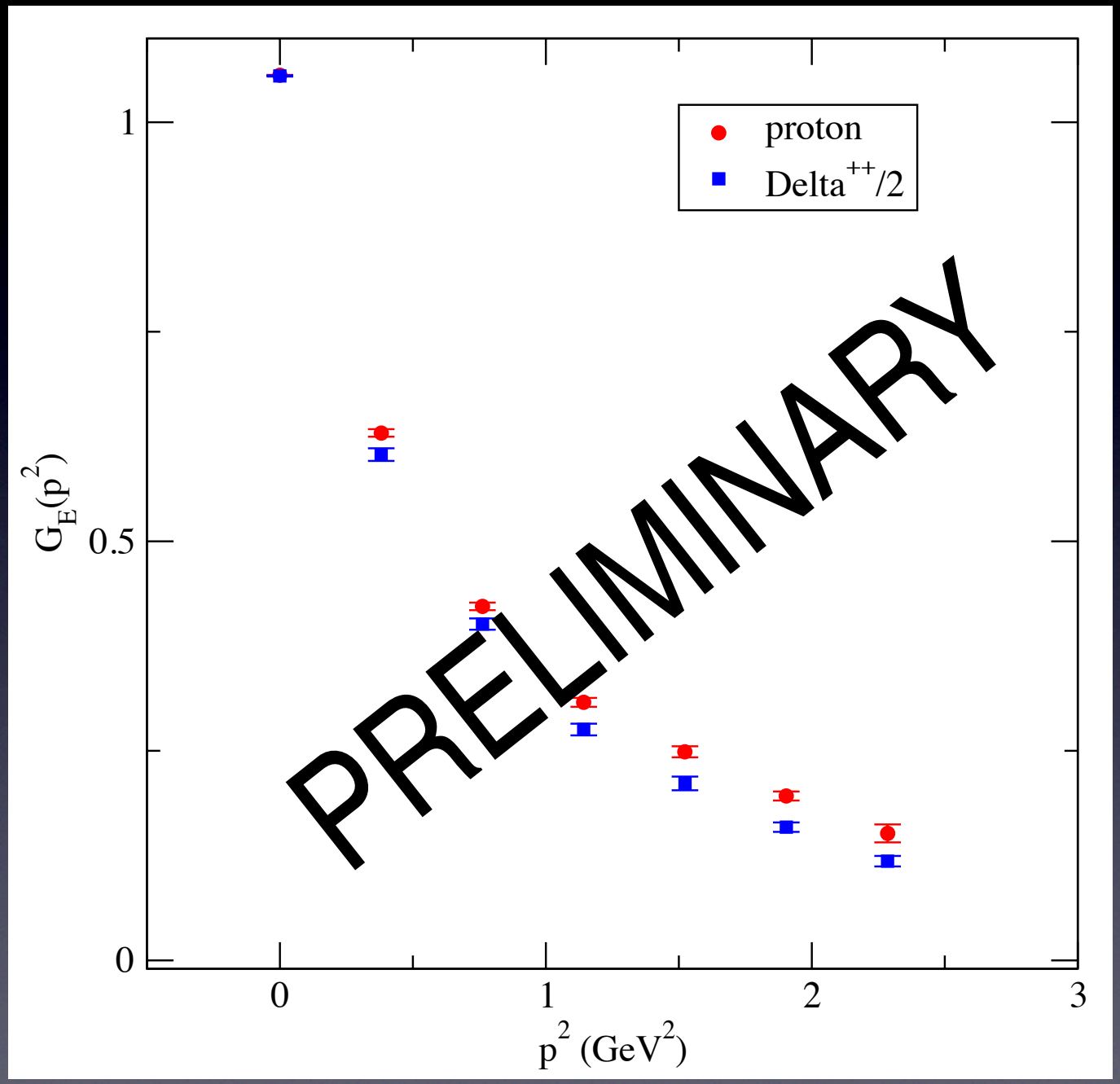


$$\frac{1}{Z_V} = 1.05(1)$$

Not unity because we're using the local current

Errors are *highly* underestimated

361 configs



The problem: excited state contamination

Are we at large enough time separation
to only see ground state?

Yes for zero momentum but questionable
for higher momenta...

Obligatory slide of lattice numbers given in unphysical units

Anisotropic Clover Lattices (via JLab)

$$\frac{a_s}{a_t} \approx 3.5 \qquad a_t^{-1} \approx 5.5 \text{ GeV}$$

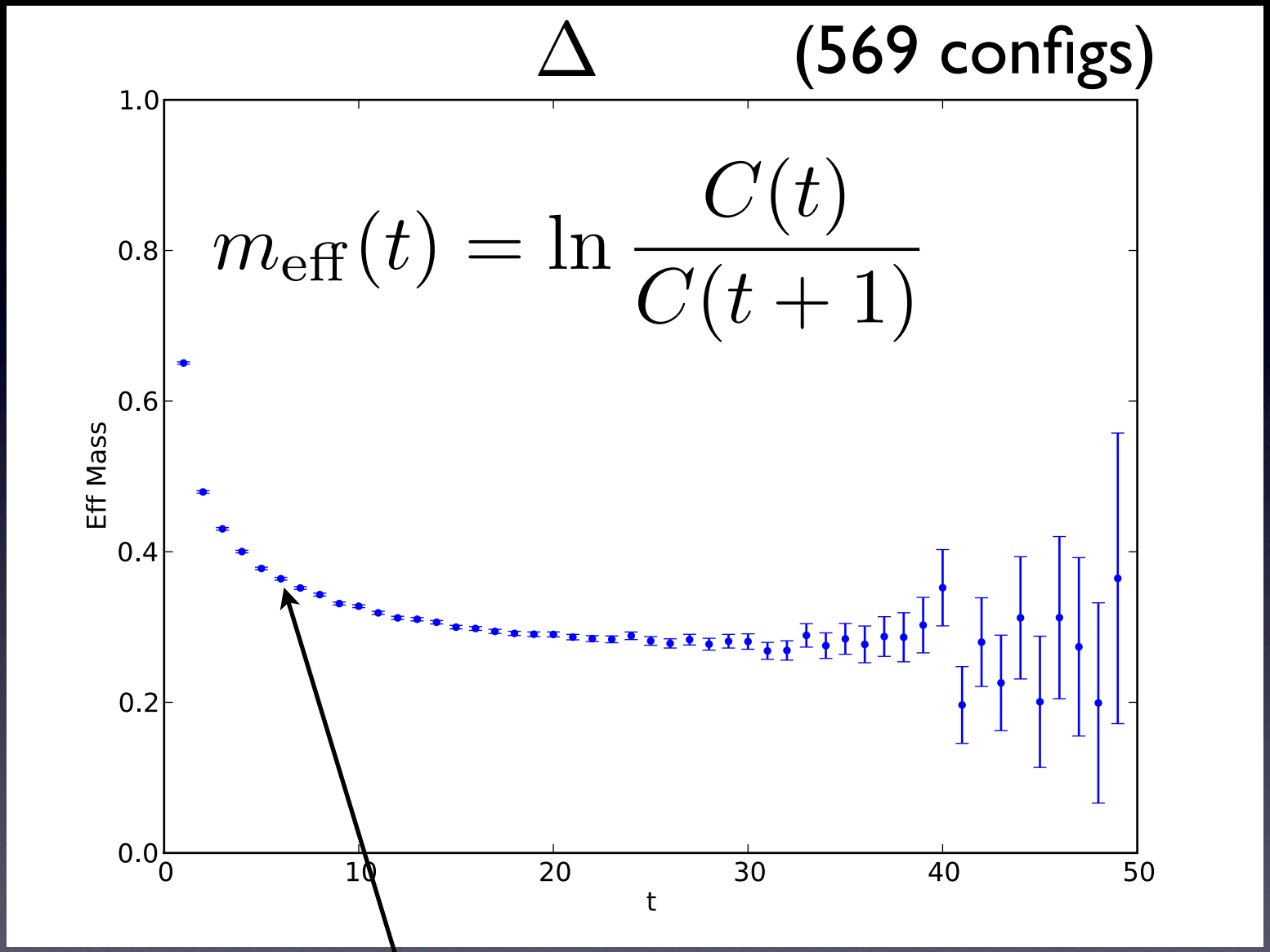
Volumes: $(16^3, 20^3, 24^3) \times 128$

$$390 \text{ (MeV)} \leq m_\pi$$

$$m_\Delta \approx 1.4 - 1.5 \text{ GeV}$$

Anisotropy allows for better resolution in time
(Good for excited states and baryons)

The problem also shows itself in "effective mass" plots



Excited state contamination

There are many operators that can be used

The only requirement is that they have
a good overlap with the state of interest

There are many operators that can be used

The only requirement is that they have
a good overlap with the state of interest

One approach is to enumerate a large number
of operators, and calculate a matrix of correlators

$$C_{ij}(t) = \langle 0|O_i(t)O_j(0)|0\rangle$$

$$C_{ij}(t) = \langle 0|O_i(t)O_j(0)|0\rangle$$

$$C_{ij}(t) = \langle 0|O_i(t)O_j(0)|0\rangle$$

Solve the Generalized Eigenvalue Problem

$$C(t)x = \lambda(t)C(t_0)x$$

$$C_{ij}(t) = \langle 0|O_i(t)O_j(0)|0\rangle$$

Solve the Generalized Eigenvalue Problem

$$C(t)x = \lambda(t)C(t_0)x$$

One can show that the eigenvalues behave like

$$\lambda_i(t, t_0) \sim e^{-m_i(t-t_0)} + \cdots$$

$$C_{ij}(t) = \langle 0|O_i(t)O_j(0)|0\rangle$$

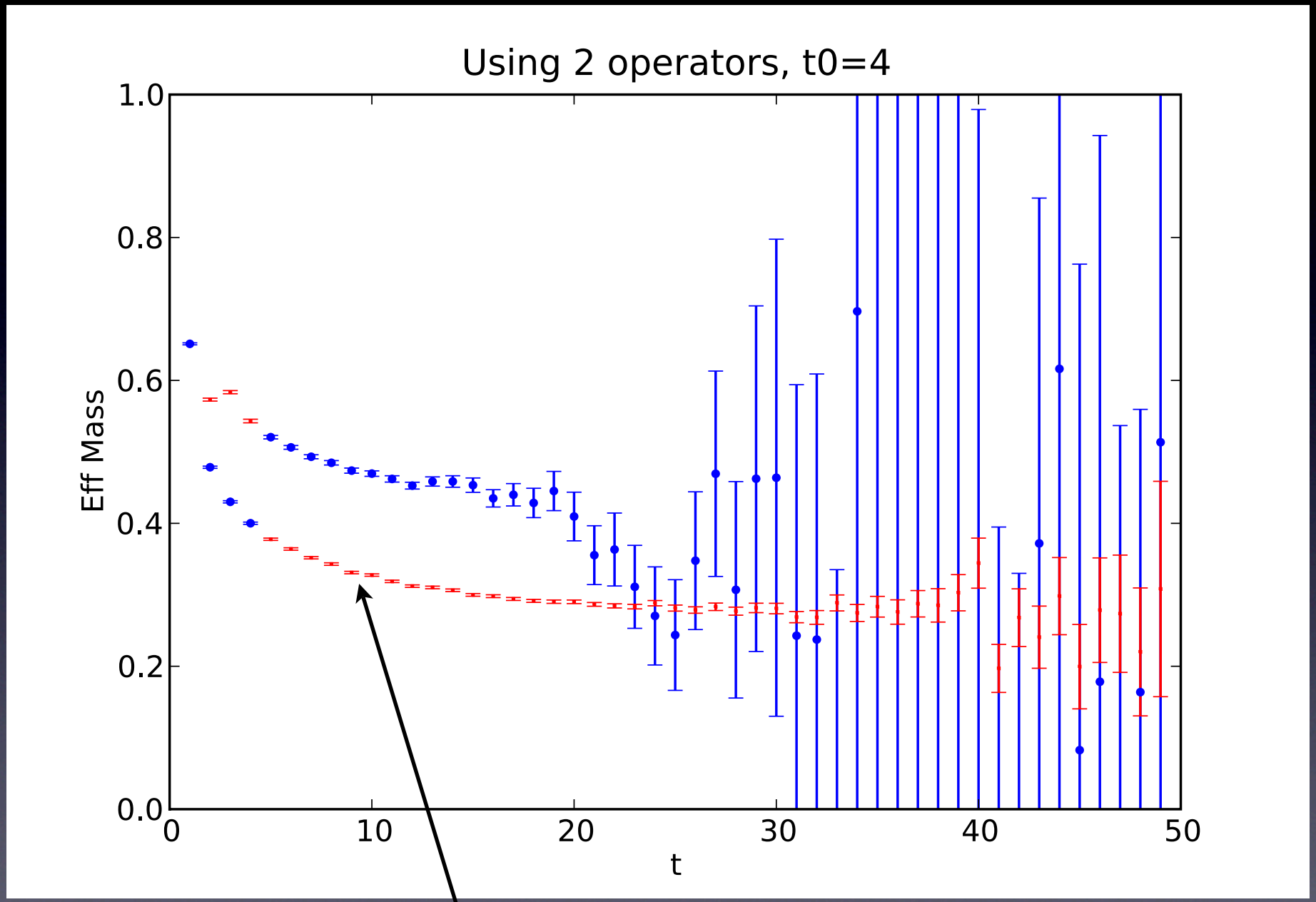
Solve the Generalized Eigenvalue Problem

$$C(t)x = \lambda(t)C(t_0)x$$

One can show that the eigenvalues behave like

$$\lambda_i(t, t_0) \sim e^{-m_i(t-t_0)} + \cdots$$

t_0 should be chosen such that $C(t_0)$ contains (ideally) all of the states in the correlator, no more, no less



Ground state has same behavior

Generalized PoF Method

Hua, Sakar (1989)
Sarkar, Pereira (1995)

Generalized PoF Method

Hua, Sakar (1989)
Sarkar, Pereira (1995)

The main point to note is that if

$$O_\Delta(t)$$

is an interpolating operator for a Δ

Generalized PoF Method

Hua, Sakar (1989)
Sarkar, Pereira (1995)

The main point to note is that if

$$O_\Delta(t)$$

is an interpolating operator for a Δ

then so is

$$O^\tau_\Delta(t) \equiv e^{H\tau} O_\Delta(t) e^{-H\tau}$$

Using this, we can consider

$$O^{\tau}_{\Delta}(t), O_{\Delta}(t)$$

distinct operators and make a matrix of correlators from a single two-point function:

Using this, we can consider

$$O^{\tau}_{\Delta}(t), O_{\Delta}(t)$$

distinct operators and make a matrix of correlators from a single two-point function:

$$C(t) = \langle 0 | O_{\Delta}(t) O^{\dagger}_{\Delta}(0) | 0 \rangle$$

Using this, we can consider

$$O^{\tau}_{\Delta}(t), O_{\Delta}(t)$$

distinct operators and make a matrix of correlators from a single two-point function:

$$C(t) = \langle 0|O_{\Delta}(t)O^{\dagger}_{\Delta}(0)|0\rangle$$

$$\begin{aligned} C(t+\tau) &= \langle 0|O^{\tau}_{\Delta}(t)O^{\dagger}_{\Delta}(0)|0\rangle \\ &= \langle 0|O_{\Delta}(t+\tau)O^{\dagger}_{\Delta}(0)|0\rangle \end{aligned}$$

Using this, we can consider

$$O^{\tau}_{\Delta}(t), O_{\Delta}(t)$$

distinct operators and make a matrix of correlators from a single two-point function:

$$C(t) = \langle 0|O_{\Delta}(t)O^{\dagger}_{\Delta}(0)|0\rangle$$

$$\begin{aligned} C(t+\tau) &= \langle 0|O^{\tau}_{\Delta}(t)O^{\dagger}_{\Delta}(0)|0\rangle \\ &= \langle 0|O_{\Delta}(t+\tau)O^{\dagger}_{\Delta}(0)|0\rangle \\ C(t+2\tau) &= \langle 0|O^{\tau}_{\Delta}(t)(O^{\tau}_{\Delta})^{\dagger}(0)|0\rangle \\ &= \langle 0|O_{\Delta}(t+2\tau)O^{\dagger}_{\Delta}(0)|0\rangle \end{aligned}$$

In fact, we could do this as much as we want

$$\begin{pmatrix} C(t) & C(t+\tau) & C(t+2\tau) & \cdots & C(t+n\tau) \\ C(t+\tau) & C(t+2\tau) & C(t+3\tau) & \cdots & C(t+(n+1)\tau) \\ C(t+2\tau) & C(t+3\tau) & C(t+4\tau) & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C(t+n\tau) & C(t+(n+1)\tau) & C(t+(n+2)\tau) & \cdots & C(t+2n\tau) \end{pmatrix}$$

Where *n* is the number of shifts we perform and τ is the amount by which we shift

In fact, we could do this as much as we want

$$\begin{pmatrix} C(t) & C(t+\tau) & C(t+2\tau) & \cdots & C(t+n\tau) \\ C(t+\tau) & C(t+2\tau) & C(t+3\tau) & \cdots & C(t+(n+1)\tau) \\ C(t+2\tau) & C(t+3\tau) & C(t+4\tau) & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C(t+n\tau) & C(t+(n+1)\tau) & C(t+(n+2)\tau) & \cdots & C(t+2n\tau) \end{pmatrix}$$

Where *n* is the number of shifts we perform and τ is the amount by which we shift

In principle, we could choose $n = T/2, \ \tau = 1$

with *T* the lattice time length

In fact, we could do this as much as we want

$$\begin{pmatrix} C(t) & C(t+\tau) & C(t+2\tau) & \cdots & C(t+n\tau) \\ C(t+\tau) & C(t+2\tau) & C(t+3\tau) & \cdots & C(t+(n+1)\tau) \\ C(t+2\tau) & C(t+3\tau) & C(t+4\tau) & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C(t+n\tau) & C(t+(n+1)\tau) & C(t+(n+2)\tau) & \cdots & C(t+2n\tau) \end{pmatrix}$$

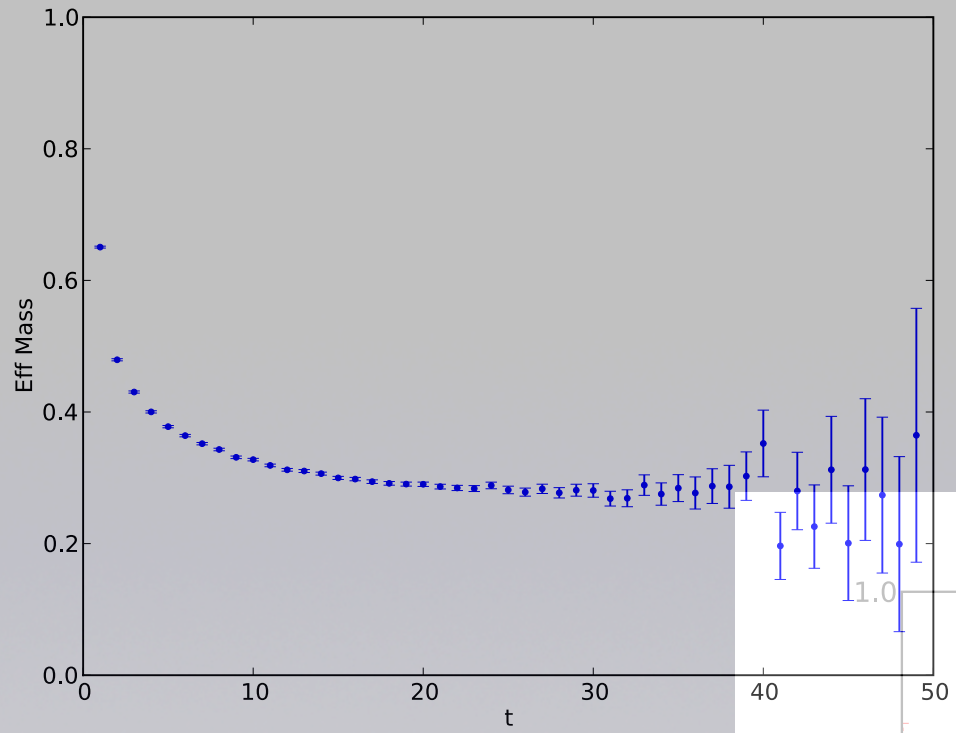
Where *n* is the number of shifts we perform and τ is the amount by which we shift

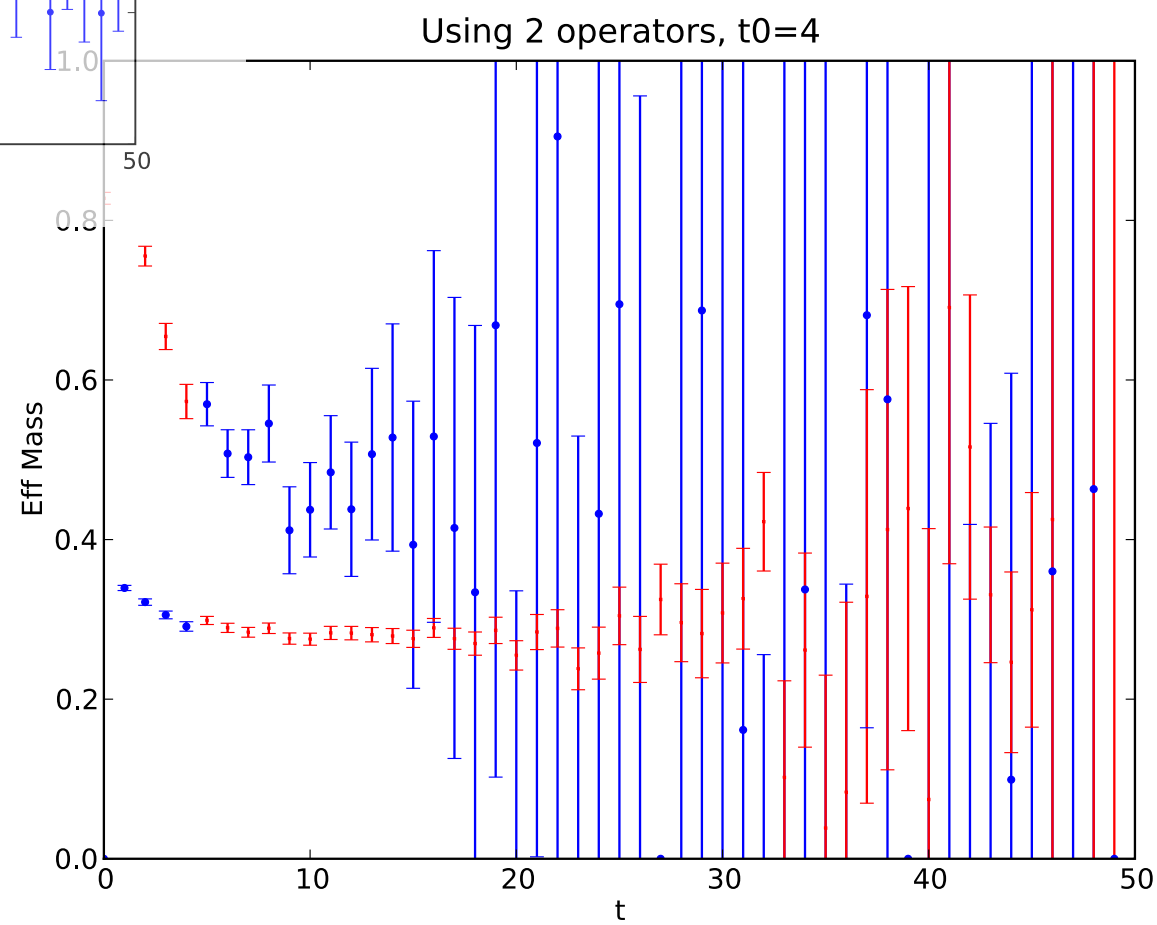
In principle, we could choose $n = T/2, \ \tau = 1$

with *T* the lattice time length

Large correlations and noise and linear dependence make this a bad idea

Same single operator as before





n, τ	m $[Q]$	fit range
$0, 0$	$0.2770(50)$ $[0.78]$	$(29, 40)$
$1, 4$	$0.2825(17)$ $[0.84]$	$(5, 28)$
$2, 4$	$0.2823(17)$ $[0.87]$	$(5, 28)$
$2, 2$	$0.2838(15)$ $[0.74]$	$(5, 30)$
$2, 8$	$0.2799(29)$ $[0.90]$	$(5, 22)$

Can we do better with more operators?

Not with only local operators...

As for the three-point correlators

$$\mathbf{C}^{3pt}(t_i, t, t_f) = \begin{pmatrix} C^{3pt}(t_i, t, t_f) & C^{3pt}(t_i, t, t_f + \tau) \\ C^{3pt}(t_i, t + \tau, t_f + \tau) & C^{3pt}(t_i, t + \tau, t_f + 2\tau) \end{pmatrix}$$

So this requires using three different sink locations (factor of three in cost, unlike the two-point correlators, as that was free)

As for the three-point correlators

$$\mathbf{C}^{3pt}(t_i, t, t_f) = \begin{pmatrix} C^{3pt}(t_i, t, t_f) & C^{3pt}(t_i, t, t_f + \tau) \\ C^{3pt}(t_i, t + \tau, t_f + \tau) & C^{3pt}(t_i, t + \tau, t_f + 2\tau) \end{pmatrix}$$

So this requires using three different sink locations (factor of three in cost, unlike the two-point correlators, as that was free)

Once the two-point correlator matrix is diagonalized with vectors V:

$$\mathbf{C}^{3pt}_{\text{diag}}(t_i, t, t_f) = V^{-1}\mathbf{C}^{3pt}(t_i, t, t_f)V$$

Conclusions

Using Pencil-of-Function techniques is much better for getting the ground state than using multiple operators

We have a good determination of $G_E(q^2)$, and need to extract other form factors (and do so on larger volumes)

Hopefully applying GPoF techniques to the three-point correlators will improve the signal even more (and systematics)